Exchangeability[1]

Sociol 114

[1] Some material in this lecture draws on past materials by Sam Wang at Cornell University. Thanks Sam!

Learning goals for today

At the end of class, you will be able to:

1. Explain exchangeability in randomized experiments
2. Make arguments about why exchangeability may not hold in particular cases when a treatment is not randomized

Exchangeable sampling from a population

Exchangeable sampling from a population

Population Outcomes

$Y_{\textbf{Maria}}$

$Y_{\textbf{William}}$

$Y_{\textbf{Rich}}$

$Y_{\textbf{Sarah}}$

$Y_{\textbf{Alondra}}$

$Y_{\textbf{Jesús}}$

Exchangeable sampling from a population

Population Outcomes	Randomized Sampling
$Y_{\textbf{Maria}}$	$S_{\textbf{Maria}} = 1$
$Y_{\textbf{William}}$	$S_{\textbf{William}} = 0$
$Y_{\textbf{Rich}}$	$S_{\textbf{Rich}} = 0$
$Y_{\textbf{Sarah}}$	$S_{\textbf{Sarah}} = 1$
$Y_{\textbf{Alondra}}$	$S_{\textbf{Alondra}} = 0$
$Y_{\textbf{Jesús}}$	$S_{\textbf{Jesús}} = 1$

Exchangeable sampling from a population

Population Outcomes	Randomized Sampling	Sampled Outcomes
Y_{Maria}	$S_{\text{Maria}} = 1$	Y_{Maria}
Y_{William}	$S_{\text{William}} = 0$	
Y_{Rich}	$S_{\text{Rich}} = 0$	
Y_{Sarah}	$S_{\text{Sarah}} = 1$	Y_{Sarah}
Y_{Alondra}	$S_{\text{Alondra}} = 0$	
$Y_{\text{Jesús}}$	$S_{\text{Jesús}} = 1$	$Y_{\text{Jesús}}$

Exchangeable sampling from a population

Population Outcomes	Randomized Sampling	Sampled Outcomes
Y_{Maria}	$S_{\text{Maria}} = 1$	Y_{Maria}
Y_{William}	$S_{\text{William}} = 0$	
Y_{Rich}	$S_{\text{Rich}} = 0$	
Y_{Sarah}	$S_{\text{Sarah}} = 1$	Y_{Sarah}
Y_{Alondra}	$S_{\text{Alondra}} = 0$	
$Y_{\text{Jesús}}$	$S_{\text{Jesús}} = 1$	$Y_{\text{Jesús}}$

Estimator:
Estimate the population mean by the sample mean

Key assumption:
Sampled and unsampled units are **exchangeable** due to random sampling

$Y \perp\!\!\!\perp S$

Now suppose our population all participate
in a randomized experiment with
treatment ($A = 1$) and control ($A = 0$) conditions

Exchangeable treatment assignment

Population Potential Outcomes

$Y^1_{\textbf{Maria}}$
$Y^1_{\textbf{William}}$
$Y^1_{\textbf{Rich}}$
$Y^1_{\textbf{Sarah}}$
$Y^1_{\textbf{Alondra}}$
$Y^1_{\textbf{Jesús}}$

Exchangeable treatment assignment

Population Potential Outcomes	**Randomized Treatment**
Y^1_{Maria}	$A_{\text{Maria}} = 1$
Y^1_{William}	$A_{\text{William}} = 0$
Y^1_{Rich}	$A_{\text{Rich}} = 0$
Y^1_{Sarah}	$A_{\text{Sarah}} = 1$
Y^1_{Alondra}	$A_{\text{Alondra}} = 0$
$Y^1_{\text{Jesús}}$	$A_{\text{Jesús}} = 1$

Exchangeable treatment assignment

Population Potential Outcomes	Randomized Treatment	Observed Outcomes
Y^1_{Maria}	$A_{\text{Maria}} = 1$	Y^1_{Maria}
Y^1_{William}	$A_{\text{William}} = 0$	
Y^1_{Rich}	$A_{\text{Rich}} = 0$	
Y^1_{Sarah}	$A_{\text{Sarah}} = 1$	Y^1_{Sarah}
Y^1_{Alondra}	$A_{\text{Alondra}} = 0$	
$Y^1_{\text{Jesús}}$	$A_{\text{Jesús}} = 1$	$Y^1_{\text{Jesús}}$

Exchangeable treatment assignment

Population Potential Outcomes	Randomized Treatment	Observed Outcomes
Y^1_{Maria}	$A_{\text{Maria}} = 1$	Y^1_{Maria}
Y^1_{William}	$A_{\text{William}} = 0$	
Y^1_{Rich}	$A_{\text{Rich}} = 0$	
Y^1_{Sarah}	$A_{\text{Sarah}} = 1$	Y^1_{Sarah}
Y^1_{Alondra}	$A_{\text{Alondra}} = 0$	
$Y^1_{\text{Jesús}}$	$A_{\text{Jesús}} = 1$	$Y^1_{\text{Jesús}}$

Estimator:
Estimate the population mean $\text{E}(Y^1)$ by the untreated sample mean

Key assumption:
Treated and untreated units are **exchangeable** due to random treatment assignment

$Y^1 \perp\!\!\!\perp A$

Exchangeable treatment assignment

Population Potential Outcomes	Randomized Treatment	Observed Outcomes
$Y^0_{\textbf{Maria}}$	$A_{\textbf{Maria}} = 1$	
$Y^0_{\textbf{William}}$	$A_{\textbf{William}} = 0$	$Y^0_{\textbf{William}}$
$Y^0_{\textbf{Rich}}$	$A_{\textbf{Rich}} = 0$	$Y^0_{\textbf{Rich}}$
$Y^0_{\textbf{Sarah}}$	$A_{\textbf{Sarah}} = 1$	
$Y^0_{\textbf{Alondra}}$	$A_{\textbf{Alondra}} = 0$	$Y^0_{\textbf{Alondra}}$
$Y^0_{\textbf{Jesús}}$	$A_{\textbf{Jesús}} = 1$	

Estimator:
Estimate the population mean $\mathrm{E}(Y^0)$ by the untreated sample mean

Key assumption:
Treated and untreated units are **exchangeable** due to random treatment assignment

$$Y^0 \perp\!\!\!\perp A$$

Exchangeable treatment assignment

Population Potential Outcomes		Randomized Treatment	Observed Outcomes	
Y^1_{Maria}	Y^0_{Maria}	$A_{\text{Maria}} = 1$	Y^1_{Maria}	
Y^1_{William}	Y^0_{William}	$A_{\text{William}} = 0$		Y^0_{William}
Y^1_{Rich}	Y^0_{Rich}	$A_{\text{Rich}} = 0$		Y^0_{Rich}
Y^1_{Sarah}	Y^0_{Sarah}	$A_{\text{Sarah}} = 1$	Y^1_{Sarah}	
Y^1_{Alondra}	Y^0_{Alondra}	$A_{\text{Alondra}} = 0$		Y^0_{Alondra}
$Y^1_{\text{Jesús}}$	$Y^0_{\text{Jesús}}$	$A_{\text{Jesús}} = 1$	$Y^1_{\text{Jesús}}$	

Exchangeable treatment assignment

Causal Estimand:

(expected outcome if assigned to treatment)
− (expected outcome if assigned to control)

$$\mathrm{E}\left(Y^1\right) - \mathrm{E}\left(Y^0\right)$$

Exchangeability Assumption:
Potential outcomes are independent of treatment assignment

$$\{Y^0, Y^1\} \perp\!\!\!\perp A$$

Empirical Estimand:

(expected outcome among the treated)
− (expected outcome among the untreated)

$$\mathrm{E}(Y \mid A = 1) - \mathrm{E}(Y \mid A = 0)$$

Exchangeable treatment assignment: Proof

$$
\begin{aligned}
&\mathrm{E}\left(Y^1\right) - \mathrm{E}\left(Y^0\right) \\
&= \mathrm{E}\left(Y^1 \mid A = 1\right) - \mathrm{E}\left(Y^0 \mid A = 0\right) \\
&= \mathrm{E}\left(Y \mid A = 1\right) - \mathrm{E}\left(Y \mid A = 0\right)
\end{aligned}
$$

Exchangeable treatment assignment: Proof

$$
\begin{aligned}
&\mathsf{E}\left(Y^1\right) - \mathsf{E}\left(Y^0\right) \\
&= \mathsf{E}\left(Y^1 \mid A = 1\right) - \mathsf{E}\left(Y^0 \mid A = 0\right) \\
&= \mathsf{E}\left(Y \mid A = 1\right) - \mathsf{E}\left(Y \mid A = 0\right) \qquad \text{by consistency}
\end{aligned}
$$

Exchangeable treatment assignment: Proof

$$
\begin{aligned}
&\mathrm{E}\left(Y^1\right) - \mathrm{E}\left(Y^0\right) \\
&= \mathrm{E}\left(Y^1 \mid A = 1\right) - \mathrm{E}\left(Y^0 \mid A = 0\right) \quad \text{by exchangeability} \\
&= \mathrm{E}\left(Y \mid A = 1\right) - \mathrm{E}\left(Y \mid A = 0\right) \qquad \text{by consistency}
\end{aligned}
$$

Exchangeable treatment assignment: Proof

$$
\begin{aligned}
&\mathrm{E}\left(Y^1\right) - \mathrm{E}\left(Y^0\right) \\
&= \mathrm{E}\left(Y^1 \mid A = 1\right) - \mathrm{E}\left(Y^0 \mid A = 0\right) \quad \text{by exchangeability} \\
&= \mathrm{E}\left(Y \mid A = 1\right) - \mathrm{E}\left(Y \mid A = 0\right) \qquad\qquad \text{by consistency}
\end{aligned}
$$

This is an example of **causal identification**:
using assumptions to arrive at an empirical quantity
(involving only factual random variables, no potential outcomes)
that equals our causal estimand if the assumptions hold

The causal estimand $\mathrm{E}(Y^1) - \mathrm{E}(Y^0)$ is **identified** by the empirical estimand $\mathrm{E}(Y \mid A = 1) - \mathrm{E}(Y \mid A = 0)$

Potential outcome exercise: Covid vaccines

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Suppose we know the following pieces of information:

- Martha was vaccinated against Covid.
  Martha tested negative 6 months later.
- Ezra was not vaccinated against Covid.
  Ezra tested positive 6 months later.

Potential outcome exercise: Covid vaccines

Suppose we know the following pieces of information:

- Martha was vaccinated against Covid.
  Martha tested negative 6 months later.
- Ezra was not vaccinated against Covid.
  Ezra tested positive 6 months later.

Which cells have known values? What are the values?

	A_i	Y_i	$Y_i^{\text{Vaccinated}}$	$Y_i^{\text{Unvaccinated}}$
Martha				
Ezra				

Experiments vs observational studies

Suppose we gathered data by surveying individuals in Fall of 2021

- Vaccinated for covid ($A_i = 1$) or not ($A_i = 0$)
- Tested positive for Covid in 2021: yes ($Y_i = 1$) or no ($Y_i = 0$)

Experiments vs observational studies

We observe evidence

- Of the individuals who are **vaccinated** ($A_i = 1$), 50% had a positive Covid test ($Y_i = 1$)
- Of the individuals who are **not vaccinated** ($A_i = 0$), 70% had a positive Covid test ($Y_i = 1$)

Experiments vs observational studies

We observe evidence

- Of the individuals who are **vaccinated** ($A_i = 1$), 50% had a positive Covid test ($Y_i = 1$)
- Of the individuals who are **not vaccinated** ($A_i = 0$), 70% had a positive Covid test ($Y_i = 1$)

How might a vaccine skeptic explain the data?

Experiments vs observational studies

Experiment designed by Pfizer **randomly assign** each individual (43,000 total) into two groups[2]:

- Two doses of BNT162b2 vaccine 21 days apart
- Two doses of placebo 21 days apart
- Whether a positive Covid test was recorded between 7 days and 14 weeks after the injection

[2] Polack et. al. NEJM 2020

Experiments vs observational studies

Experiment designed by Pfizer **randomly assign** each individual (43,000 total) into two groups[2]:

- Two doses of BNT162b2 vaccine 21 days apart
- Two doses of placebo 21 days apart
- Whether a positive Covid test was recorded between 7 days and 14 weeks after the injection

- Of the individuals who were given the vaccine ($A_i = 1$), 0.04% had a positive Covid test ($Y_i = 1$)
- Of the individuals who were given the placebo ($A_i = 0$), 0.9% had a positive Covid test ($Y_i = 1$)
- Individuals who received the placebo were $\approx$ 20 times more likely to get Covid

[2] Polack et. al. NEJM 2020

Experiments vs observational studies

Experiment designed by Pfizer **randomly assign** each individual (43,000 total) into two groups[2]:

- Two doses of BNT162b2 vaccine 21 days apart
- Two doses of placebo 21 days apart
- Whether a positive Covid test was recorded between 7 days and 14 weeks after the injection

- Of the individuals who were given the vaccine ($A_i = 1$), 0.04% had a positive Covid test ($Y_i = 1$)
- Of the individuals who were given the placebo ($A_i = 0$), 0.9% had a positive Covid test ($Y_i = 1$)
- Individuals who received the placebo were ≈ 20 times more likely to get Covid

Do the skeptic's objections still hold?

[2] Polack et. al. NEJM 2020

Why experiments “work”: Exchangeability

Table 1. Demographic Characteristics of the Participants in the Main Safety Population.*

Characteristic	BNT162b2 (N=18,860)	Placebo (N=18,846)	Total (N=37,706)
Sex — no. (%)			
Male	9,639 (51.1)	9,436 (50.1)	19,075 (50.6)
Female	9,221 (48.9)	9,410 (49.9)	18,631 (49.4)
Race or ethnic group — no. (%)†			
White	15,636 (82.9)	15,630 (82.9)	31,266 (82.9)
Black or African American	1,729 (9.2)	1,763 (9.4)	3,492 (9.3)
Asian	801 (4.2)	807 (4.3)	1,608 (4.3)
Native American or Alaska Native	102 (0.5)	99 (0.5)	201 (0.5)
Native Hawaiian or other Pacific Islander	50 (0.3)	26 (0.1)	76 (0.2)
Multiracial	449 (2.4)	406 (2.2)	855 (2.3)
Not reported	93 (0.5)	115 (0.6)	208 (0.6)
Hispanic or Latinx	5,266 (27.9)	5,277 (28.0)	10,543 (28.0)
Country — no. (%)			
Argentina	2,883 (15.3)	2,881 (15.3)	5,764 (15.3)
Brazil	1,145 (6.1)	1,139 (6.0)	2,284 (6.1)
South Africa	372 (2.0)	372 (2.0)	744 (2.0)
United States	14,460 (76.7)	14,454 (76.7)	28,914 (76.7)
Age group — no. (%)			
16–55 yr	10,889 (57.7)	10,896 (57.8)	21,785 (57.8)
>55 yr	7,971 (42.3)	7,950 (42.2)	15,921 (42.2)
Age at vaccination — yr			
Median	52.0	52.0	52.0
Range	16–89	16–91	16–91
Body-mass index‡			
≥30.0: obese	6,556 (34.8)	6,662 (35.3)	13,218 (35.1)

* Percentages may not total 100 because of rounding.
† Race or ethnic group was reported by the participants.
‡ The body-mass index is the weight in kilograms divided by the square of the height in meters.

Why experiments "work": Exchangeability

In random experiments, the distribution of **potential outcomes** for those who are treated and those who are not treated (control group) are identically distributed!

$$\{Y^1, Y^0\} \perp\!\!\!\perp A$$

This is **exchangeability**

Question. Does exchangeability imply $Y \perp\!\!\!\perp A$?

Why experiments "work": Exchangeability

Exchangeability is about **potential** rather than **observed** outcomes

$$\{Y^0, Y^1\} \perp\!\!\!\perp A \qquad \text{rather than} \qquad Y \not\!\perp\!\!\!\perp A$$

Why experiments "work": Exchangeability

Exchangeability is about **potential** rather than **observed** outcomes

$$\{Y^0, Y^1\} \perp\!\!\!\perp A \qquad \text{rather than} \qquad Y \not\!\perp\!\!\!\perp A$$

- Potential outcomes are independent of treatment $\{Y^0, Y^1\} \perp\!\!\!\perp A$
  - Example: Risk of covid under no vaccine (Y^0) is the same for those with and without a vaccine

Why experiments "work": Exchangeability

Exchangeability is about **potential** rather than **observed** outcomes

$$\{Y^0, Y^1\} \perp\!\!\!\perp A \qquad \text{rather than} \qquad Y \not\perp\!\!\!\perp A$$

- Potential outcomes are independent of treatment $\{Y^0, Y^1\} \perp\!\!\!\perp A$
  - Example: Risk of covid under no vaccine (Y^0) is the same for those with and without a vaccine
- Observed outcomes are not independent of treatment $Y \not\perp\!\!\!\perp A$
  - Example: Risk of covid is lower for those with the vaccine
  - Why? Because for them $Y = Y^1$. For others, $Y = Y^0$.
  - If A affects Y, then $Y \not\perp\!\!\!\perp A$

Why experiments “work”: Exchangeability

Exchangeability is about **potential** rather than **observed** outcomes

$$\{Y^0, Y^1\} \perp\!\!\!\perp A \qquad \text{rather than} \qquad Y \not\perp\!\!\!\perp A$$

- Potential outcomes are independent of treatment $\{Y^0, Y^1\} \perp\!\!\!\perp A$
  - Example: Risk of covid under no vaccine (Y^0) is the same for those with and without a vaccine
- Observed outcomes are not independent of treatment $Y \not\perp\!\!\!\perp A$
  - Example: Risk of covid is lower for those with the vaccine
  - Why? Because for them $Y = Y^1$. For others, $Y = Y^0$.
  - If A affects Y, then $Y \not\perp\!\!\!\perp A$

Under exchangeability, the only reason $Y \not\perp\!\!\!\perp A$ is if A causes Y.

Design a hypothetical experiment

- Define a treatment and an outcome
- Design a randomized experiment
  - Who would you enroll?
  - How would you randomize the treatment?
  - When and how would you measure the outcome?
- Think of a criticism that could be levied against you if you had not randomized the treatment, which is overcome by randomization

Learning goals for today

At the end of class, you will be able to:

1. Explain exchangeability in randomized experiments
2. Make arguments about why exchangeability may not hold in particular cases when a treatment is not randomized